

FOLIATION BY CONSTANT MEAN CURVATURE SPHERES ON ASYMPTOTICALLY FLAT MANIFOLDS

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§0 Introduction.

The main result of this paper is the following.

Main Theorem. *Let M^{n+1} be an asymptotically flat manifold, $n \geq 2$, and Ω an end of M having nonzero mass. Then there is on Ω a smooth codimension one foliation \mathcal{F}_o by constant mean curvature spheres. \mathcal{F}_o is balanced and regular at ∞ . Moreover, it is the unique weakly balanced and regular C^2 foliation on Ω by closed hypersurfaces of constant mean curvature.*

“Balanced at ∞ ” means that near ∞ the leaves approach geodesic spheres of a fixed center. “Weakly balanced” roughly means that the “geodesic centers” of the leaves do not shift to ∞ as fast as the farthest points on the leaves. “Regular at ∞ ” means that the rescaled second fundamental form of the leaves is uniformly bounded. For the precise definitions we refer to the next section. Note that \mathcal{F}_o actually foliates $\overline{\Omega \setminus K}$ for a compact region K . In the statement of the theorem the phrase “on Ω ” is used in a more general way than standard. For the precise meaning of uniqueness we refer to the Uniqueness Theorem in §2.

We obtained the existence part of this result (and a somewhat weaker uniqueness result) at the end of 1988. (In [Ye1], we showed that around a nondegenerate critical point of the scalar curvature function in a Riemannian manifold, there exists a unique regular foliation by constant mean curvature spheres. We mentioned that the arguments extend to yield the existence result stated in the Main Theorem.) Then we obtained the uniqueness part of the Main Theorem. Recently there has been more interest in this problem and several colleagues have inquired about the details of the proof of this result. Also recently, we showed in [Ye2] that in dimension 3 all *diameter-pinched* (see Definition 5) C^2 foliations by constant mean curvature

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spheres are regular. Since “diameter-pinched” implies “weakly balanced”, the Main Theorem implies the following

Strong Uniqueness Theorem. *Let M be a 3-dimensional asymptotically flat manifold, then on each end of nonzero mass of M there is a unique diameter-pinched C^2 foliation by constant mean curvature spheres.*

For details we refer to [Ye2]. We remark that the “diameter-pinched” condition is (much) weaker than the “balanced” condition.

Asymptotically flat manifolds arise in general relativity. The significance of the above results is to provide a canonical and regular geometric structure on asymptotically flat manifolds. Though the usual concept of asymptotical flatness is sufficient for applications, it is expressed in terms of coordinates and hence is not geometrically canonical. (This is a problem on S. T. Yau’s list [Ya] of open problems in differential geometry. Note that in [B] a good understanding of the asymptotical coordinates is provided.) The canonical geometric structure provided by the above results serves to make the concept of asymptotical flatness more geometrical and canonical. In particular, it immediately makes the mass a geometric invariant. It also serves as a geometric linkage between different asymptotical coordinates by the way of its construction. Indeed, it should be possible to completely characterize asymptotically flat ends of nonzero mass in terms of balanced and regular foliations by constant mean curvature spheres. One also expects further applications in general relativity. A philosophical implication is a concept of “center of universe” which may be defined as the region surrounded by the foliations.

The program of constructing foliations by constant mean curvature spheres on asymptotically flat manifolds was initiated by S. T. Yau. (See [Ya] and [CY].) He and G. Huisken have an independent proof of existence of foliations by constant mean curvature spheres on asymptotically flat ends of *positive* mass [HY1][HY2]. They apply the mean curvature flow to deform Euclidean spheres in asymptotical coordinates. Positive mass implies a stability estimate which is employed to show convergence of the flow. On the other hand, Yau and Huisken [HY2] showed that on a 3-dimensional asymptotically flat manifold of positive mass, the constructed foliation is the unique foliation by *stable* spheres of constant mean curvature. In comparison, our existence and uniqueness results hold for both positive and negative mass and in all dimensions. But we note that our conditions on foliations for the uniqueness are rather different from that of Yau-Huisken. Their condition is stability as just mentioned. Our conditions are weak balance and regularity in general dimensions, and diameter pinching in dimension 3. These have a different flavor than Yau and Huisken’s stability uniqueness. It is unknown whether uniqueness holds without any condition. In the local Riemannian situation, we do have such a universal uniqueness result in dimension 3, see [Y2]. But the situation of asymptotically flat ends is more subtle, see the discussion below.

Now we would like to discuss the proof of the Main Theorem. There are several delicate aspects here. It is natural to attempt to perturb Euclidean spheres in asymptotical coordinates in order to construct constant mean curvature spheres.

But a naive application of the implicit function theorem does not work, because the linearized operator of the constant mean curvature equation in the problem has a nontrivial kernel. To resolve this difficulty, we apply the crucial idea in [Ye1] of moving centers. Namely we first perturb the center of the asymptotical coordinates and then perform normal perturbation of the Euclidean spheres. Both in [Ye1] and here the effect of the center perturbations is asymptotically degenerate. We remove this degeneracy by carefully expanding the equation and balancing the center perturbations against the normal perturbations. On the other hand, it is important to control the magnitude of the center perturbations in order to retain the foliation property. The situation of asymptotically flat ends is more delicate than the local picture in [Ye1], because the asymptotical flat structure deteriorates when the center is shifted too far away. This problem is even more serious for uniqueness than for existence, and it is the reason for the requirement of weak balance. A priori, a foliation by constant mean curvature spheres can differ dramatically from the one we constructed. We have to obtain strong geometric control of the leaves in order to derive uniqueness. Note that weak balance is a fairly weak geometric condition. We think that it is necessary for uniqueness.

We acknowledge interesting discussions with G. Huisken.

§1 Moving Centers and Perturbation.

Since we deal with ends of asymptotically flat manifolds, it is convenient to introduce the concept of asymptotically flat ends. An asymptotically flat manifold is then a complete Riemannian manifold which is the union of a compact region and finitely many asymptotically flat ends.

Definition 1. Let M be a complete Riemannian manifold of dimension $n+1$ with $n \geq 2$. Let g be the metric of M . A closed domain Ω of M is called an *asymptotically flat end* if there is a coordinate map from Ω to $\mathbb{R}^{n+1} \setminus \mathbb{B}_{R_o}^o(o)$ for some $R_o > 0$ such that on this coordinate chart the metric g can be written

$$(1.1) \quad g_{ij}(x) = (1 + \frac{\sigma}{r^{n-1}})\delta_{ij} + h_{ij}(x),$$

where $r = |x|$, σ is a constant and the h_{ij} 's satisfy

$$\begin{aligned} h_{ij} &= O(\frac{1}{r^n}), h_{ij,k} = O(\frac{1}{r^{n+1}}), h_{ij,k\ell} = O(\frac{1}{r^{n+2}}), \\ h_{ij,k\ell m} &= O(\frac{1}{r^{n+3}}), h_{ij,k\ell mm'} = O(\frac{1}{r^{n+4}}), \end{aligned}$$

as $r \rightarrow \infty$ ($h_{ij,k}$ etc. denote partial derivatives, e.g. $h_{ij,k} = \frac{\partial}{\partial x^k} h_{ij}$). The constant σ is called the *mass* or *energy* of Ω . (This differs from the usual definition [LP] by a dimensional factor.)

Note that more general concepts of asymptotically flat manifolds have been introduced in the literature, see e.g. [LP] and [Ye2]. But asymptotically flat manifolds as defined here are the most important. (See e.g. [SY]. For technical reasons we required decay of up to the fourth order derivatives of h_{ij} . This same condition is also assumed in [HY].) Hence we focus on them in this paper.

Definition 2. Let \mathcal{F} be a foliation of codimension 1 on an asymptotically flat end Ω , whose leaves are all closed. We say that \mathcal{F} is *balanced at ∞* or *balanced*, if there is a point $p \in \Omega$ such that

$$\frac{\text{dist}(p, S)}{\text{diam}(p, S)} \rightarrow 1 \text{ for } S \in \mathcal{F} \text{ as } \text{dist}(p, S) \rightarrow \infty,$$

where $\text{diam}(p, S) = \max_{q \in S} \text{dist}(p, q)$.

Definition 3. Let \mathcal{F} be as above. For $S \in \mathcal{F}$, let Ω_S be the open domain on the outside of S . Put

$$s(S) = \max_{p \in \Omega \setminus \Omega_S} \frac{\text{dist}(p, S)}{\text{diam}(p, S)}.$$

A point $p \in \Omega \setminus \Omega_S$ is called a *geodesic center* of S , if the maximum $s(S)$ is achieved at p . We say that \mathcal{F} is *weakly balanced*, if there are geodesic centers $p(S)$ of S such that

$$\limsup \frac{\text{dist}(p_o, p(S))}{\text{diam}(p(S), S)} < 1 \text{ for } S \in \mathcal{F} \text{ as } \text{diam}(p_o, S) \rightarrow \infty,$$

where p_o is a fixed point in Ω . We shall denote the above limit by $b(S)$.

Definition 4. Let \mathcal{F} be as above. Furthermore, assume that \mathcal{F} is of class C^2 . We say that \mathcal{F} is *regular at ∞* or *regular*, if

$$\limsup_{\text{diam}(S) \rightarrow \infty} \|A_S\|_{C^0(S)} \text{diam}(S) < \infty \text{ for } S \in \mathcal{F},$$

where A_S denotes the second fundamental form of S .

Definition 5. Let \mathcal{F} be as above. We say that \mathcal{F} is *diameter-pinched at ∞* or *diameter pinched*, if

$$\limsup \frac{\text{diam}(p_o, S)}{\text{dist}(p_o, S)} < \infty \text{ for } S \in \mathcal{F} \text{ as } \text{diam}(p_o, S) \rightarrow \infty,$$

where p_o is a fixed point in Ω .

Let Ω be an asymptotically flat end of dimension $n + 1$, $n \geq 2$, whose mass σ is nonzero. We can identify Ω with $\mathbb{R}^{n+1} \setminus \overset{o}{\mathbb{B}}_{R_o}$ for some $R_o > 0$. Without loss of generality, we assume $R_o = 1$. To construct a balanced and regular foliation by constant mean curvature spheres on Ω , we apply the moving center method in [Ye1]. Let ν denote the inward Euclidean unit normal of $S^n := \partial \mathbb{B}_1(o)$ and α_r the dilation $x \mapsto rx$ for $r > 0$. For $\varphi \in C^2(S^n)$ and $\tau \in \mathbb{R}^{n+1}$ we define $S_\varphi^n = \{x + \varphi(x)\nu(x) : x \in S^n\}$ and $S_{r,\tau,\varphi} = \alpha_r(\tau(S_\varphi^n))$, where the action of τ is defined to be the translation by τ . Note that $S_{r,\tau,0} = \partial \mathbb{B}_r(\tau)$ and S_φ^n is an embedded C^2 surface if only $\|\varphi\|_{C^1} \leq \varepsilon_o$ for some $\varepsilon_o \in (0, \frac{1}{4})$. (We use the standard metric

on S^n unless otherwise stated.) For $0 < r < \frac{1}{4}$, $\|\varphi\|_{C^1} \leq \varepsilon_o$, $\tau \in \mathbb{R}^{n+1}$ and $x \in S^n$ with $|\tau| + r + \|\varphi\|_{C^o} \leq 1$, we put

$$H(r, \tau, \varphi)(x) = \frac{1}{r} \left(\text{the inward mean curvature of the surface } S_{r, \tau, \varphi} \right. \\ \left. \text{at } \frac{1}{r}(\tau + x + \varphi(x)\nu(x)) \right),$$

where of course we use the metric g on Ω . By (1.1), it is easy to see that $H(r, \tau, \varphi)$ extends to $r = 0$ in a C^3 fashion. We are going to compute $H(r, \tau, 0)$. Fix r_o, τ_o and $x_o \in S^n$. We need to compute the mean curvature of $\partial\mathbb{B}_{1/r_o}(\tau_o/r_o)$ near $y_o = (\tau_o + x_o)/r_o$. Choose coordinate vector fields X_1, \dots, X_n on $\partial\mathbb{B}_{1/r_o}(\tau_o/r_o)$ near y_o such that they are orthonormal at y_o (with respect to the metric g). We can extend them to coordinate vector fields around y_o in the following way: $\tilde{X}_i((\tau_o + x)/r) = \frac{r_o}{r} X_i((\tau_o + x)/r_o)$. We set $Y((\tau_o + x)/r) = -x$. Then we have for $y = (\tau_o + x)/r_o$

$$(1.2) \quad \sum_{i=1}^n g(\nabla_{X_i} Y, X_i) \Big|_y = \frac{1}{2} \sum_{i=1}^n Y g(\tilde{X}_i, \tilde{X}_i) \Big|_y \\ = nr_o + \frac{1}{2} \sum_{i=1}^n Y g(X_i, X_i) \Big|_y,$$

where ∇ denotes the Levi-Civita connection and X_i is the euclidean parallel translation of the original X_i along the radial lines. But

$$\sum_{i=1}^n g(X_i, X_i) \Big|_y = g_{ii}(y) - g_{ij}(y)x^i x^j,$$

where the summation convention is used on the right hand side for $1 \leq i, j \leq n+1$. Hence

$$(1.3) \quad \sum_{i=1}^n g(\nabla_{X_i} Y, X_i) \Big|_y = nr_o + \frac{1}{2} (g_{ii,k}(y)x^k - g_{ij,k}(y)x^i x^j x^k).$$

Next let Z denote the inward unit normal of $\partial\mathbb{B}_{1/r_o}(\tau_o)$. Z can be constructed explicitly from Y by standard methods of linear algebra. Indeed, there are linearly independent vectors $v_1(y), \dots, v_n(y)$ which are linear functions of x and are orthogonal to x in the Euclidean metric. Applying the Gram-Schmidt procedure we obtain from the v_i 's orthonormal tangent vectors $w_1(y), \dots, w_n(y)$ at y . Every w_i can be written in the form $F(g_{ij}(y), x)$ for a smooth function F of t_{ij} , $1 \leq i, j \leq n+1$ and x . Finally, set $Z_1 = \sum_{i=1}^n h(Y, w_i)w_i$, where h denotes the tensor h_{ij} at y . Then

$$Z = \frac{Y - Z_1}{g(Y - Z_1, Y - Z_1)^{1/2}}.$$

We compute

$$g(Y - Z_1, Y - Z_1) = 1 + \frac{\sigma}{|y|^{n-1}} + h_{ij}Y^iY^j - 2g_{ij}Z_1^iY^j + g_{ij}Z_1^iZ_1^j,$$

whence

$$(1.4) \quad Z - Y = -\frac{\sigma}{2|y|^{n-1}}Y + \frac{1}{|y|^{2n-2}}F_1(g_{ij}, h_{ij}, |y|^{-1}, x) \\ + h_{ij}F_{ij}(g_{k\ell}, h_{k\ell}, |y|^{-1}, x)$$

with smooth functions F_1 and F_{ij} . Next we observe that the covariant derivative ∇ can be explicitly written and that we can replace X_i by w_i in (1.3). Since the mean curvature of $\partial\mathbb{B}_{1/r_o}(\tau_o)$ is given by $\sum_{i=1}^n g(\nabla_{w_i}Z, w_i)$, we deduce from (1.3) that

$$(1.5) \quad H(r, \tau, 0)(x) = n(1 - \frac{\sigma}{2|y|^{n-1}}) + \frac{1}{2r}(g_{ii,k}x^k - g_{ij,k}x^ix^jx^k) \cdot \\ (1 - \frac{\sigma}{2|y|^{n-1}}) + \frac{1}{r|y|^{n+1}}\tilde{F}_1(g_{ij}, |y|g_{ij,k}, h_{ij}, |y|h_{ij,k}, |y|^{-1}, |y|^{-1}y, x) \\ + \frac{1}{|y|^n}\tilde{F}_2(g_{ij}, h_{ij}, |y|^{-1}, x) + \frac{1}{r}h_{ij,k}F_{ijk}(g_{\ell m}, h_{\ell m}, |y|^{-1}, x) \\ + \frac{1}{r|y|}h_{ij}\tilde{F}_{ij}(g_{k\ell}, |y|g_{k\ell}, h_{k\ell}, |y|h_{k\ell}, |y|^{-1}, |y|^{-1}y, x) \\ + h_{ij}\bar{F}_{ij}(g_{k\ell}, h_{k\ell}, |y|^{-1}, x).$$

Here r_o, τ_o have been replaced by r and τ , $y = (\tau + x)/r$, the variable for $g_{ij}, g_{ij,k}$ etc. is y , and $\tilde{F}_1, \tilde{F}_2, F_{ijk}, \tilde{F}_{ij}, \bar{F}_{ij}$ are smooth functions. We have

$$(1.6) \quad g_{ij,k} = \frac{\sigma(1-n)}{|y|^{n+1}}y^k\delta_{ij} + h_{ij,k},$$

$$(1.7) \quad |y|^{-1} = r|\tau + x|^{-1} = r(1 - x^i\tau^i + \tau^i\tau^jb_{ij}(\tau, x))$$

for smooth functions b_{ij} which are defined for all $x \in S^n$ and $\tau \in \mathbb{R}^{n+1}$ with $|\tau| < 1$. From (1.5), (1.6) and (1.7) we then deduce

$$(1.8) \quad H(r, \tau, 0)(x) = n - \frac{\sigma n^2}{2}r^{n-1} + \frac{\sigma n(n-1)(n+1)}{2}r^{n-1}\tau^ix^i \\ + r^{n-1}\tau^i\tau^j\tilde{b}_{ij}(\tau, x) \\ + r^{n-1}f(r, \tau, x),$$

with smooth functions \tilde{b}_{ij} , f for $r > 0$, $|\tau| < 1$ and $x \in S^n$ such that

$$1) \|f(r, \tau, \cdot)\|_{C^1(S^n)} \leq C(|\tau|)r,$$

$$2) \|d_\tau f(r, \tau, \cdot)\|_{C^1(S^n)} \leq C(|\tau|)r,$$

$$3) \left\| \frac{\partial f}{\partial r}(r, \tau, \cdot) \right\|_{C^1(S^n)} \leq C(|\tau|)$$

for a positive continuous function $C(t)$ defined for $|t| < 1$, where $d_\tau f$ denotes the differential of f in τ . We put $\tilde{f} = f + \tilde{b}_{ij}\tau^i\tau^j$.

Our goal is to find solutions τ, φ of the equation

$$(1.9) \quad H(r, \tau, \varphi) = n - \frac{\sigma n^2}{2} r^{n-1}.$$

We consider $H(r, \tau, \cdot)$ as a mapping from $C^{2,1/2}(S^n)$ into $C^{0,1/2}(S^n)$ and let H_φ denote the differential of H w.r.t. φ . Set $g_{r,\tau} = r^2 g$. Then $H(r, \tau, \varphi)$ is the mean curvature of S^n and $H_\varphi(r, \tau, 0)$ is just the Jacobi operator $\Delta + \|A\|^2 + Rc(\bar{\nu})$ on S^n relative to the metric $g_{r,\tau}$, where Rc denotes Ricci curvature and $\bar{\nu}$ the inward unit normal of S^n . We indicate the dependence on $g_{r,\tau}$ as follows

$$(1.10) \quad H_\varphi(r, \tau, 0) = \Delta_{r,\tau} + \|A_{r,\tau}\|^2 + Rc_{r,\tau}.$$

Note that $g_{\tau,r}$ converges to the Euclidean metric in the C^4 topology as $r \rightarrow 0$. It follows that $H_\varphi(0, \tau, 0) = L := \Delta_{S^n} + n$, where Δ_{S^n} is the standard Laplace operator on S^n . Let Ker denote the kernel of L which is spanned by the Euclidean coordinate functions. We have the orthogonal L_2 -decompositions $C^{2,1/2}(S^n) = Ker \oplus Ker^\perp$ and $C^{0,1/2}(S^n) = Ker \oplus L(Ker^\perp)$. Let P denote the orthogonal projection from $C^{0,1/2}(S^n)$ onto Ker and $T : Ker \rightarrow \mathbb{R}^{n+1}$ the isomorphism sending $x^i|_{S^n}$ to \mathbf{e}_i = the i -th coordinate basis. Put $\tilde{P} = TP$. Then

$$(1.11) \quad \tilde{P}(H(r, \tau, 0)) = \frac{1}{2}n(n-1)(n+1)\sigma\omega_{n+1}r^{n-1}\tau + r^{n-1}\tilde{P}(\tilde{f}(r, \tau, \cdot)),$$

where $\omega_{n+1} = \text{vol}(\mathbb{B}_1(o))$. To solve (1.9) we introduce the following expansions

$$(1.12) \quad \begin{aligned} H(r, \tau, \varphi) &= H(r, \tau, 0) + r \int_0^1 \int_0^1 H_{\varphi r}(sr, \tau, t\varphi) \varphi ds dt \\ &\quad + \int_0^1 \int_0^1 t H_{\varphi \varphi}(0, \tau, st\varphi) \varphi \varphi ds dt \\ &\quad + L\varphi, \end{aligned}$$

where the subscript r means the partial derivative in r and

$$H_{\varphi \varphi}(r, \tau, \psi) \varphi \varphi' = \frac{d}{dt} H_\varphi(r, \tau, \psi + t\varphi') \varphi \Big|_{t=0}.$$

We first consider the equation

$$(1.13) \quad \tilde{P}(H(r, \tau, r^{n-1}\varphi)) = 0 .$$

Dividing it by r^{n-1} and applying (1.11), (1.12) we can reduce it to the following

$$(1.14) \quad \frac{1}{2}n(n-1)(n+1)\sigma\omega_{n+1}\tau + \tilde{P}(\tilde{f}(r, \tau, \cdot)) + r\tilde{P}(q(r, \tau, \varphi)) = 0,$$

where

$$\begin{aligned} q(r, \tau, \varphi) &= \int_0^1 \int_0^1 H_{\varphi r}(sr, \tau, tr^{n-1}\varphi) \varphi ds dt \\ &\quad + r^{n-1} \int_0^1 \int_0^1 tH_{\varphi\varphi}(0, \tau, str^{n-1}\varphi) \varphi \varphi ds dt. \end{aligned}$$

(Note that $\tilde{P}L = 0$.) Easy computations show that q has the following properties

- 1) $\|q(r, \tau, \varphi)\|_{C^{0,1/2}(S^n)} \leq \beta(\|\varphi\|),$
- 2) $\|d_\tau q(r, \tau, \varphi)\|_{C^0(S^n)} \leq \beta(\|\varphi\|),$
- 3) $\|q_\varphi(r, \tau, \varphi)\| \leq \beta(\|\varphi\|)$ with $q_\varphi : C^{2,1/2}(S^n) \rightarrow C^{0,1/2},$
- 4) $\|\frac{\partial q}{\partial r}(r, \tau, \varphi)\|_{C^{0,1/2}(S^n)} \leq \beta(\|\varphi\|).$

Here and in the sequel, β denotes a positive continuous function on \mathbb{R} and $\|\varphi\| = \|\varphi\|_{C^{2,1/2}(S^n)}.$

Let $Q(r, \tau, \varphi)$ denote the left hand side of (1.14). Then for each given bound on φ , $Q(r, 0, \varphi)$ approaches zero uniformly as $r \rightarrow 0$. This follows from the properties of q and \tilde{f} . Also by virtue of these properties, the differential $d_\tau Q$ is nearly the identity for small r and τ . Hence we can apply the implicit function theorem to obtain a unique solution $\tau(r, \varphi)$ of the equation (1.14) and hence (1.13) for small r which lies near zero. There holds $\tau(r, \varphi) \rightarrow 0$ as $r \rightarrow 0$. We also have the following estimates

$$\|\tau_\varphi\| \leq \beta(\|\varphi\|)r, \quad \left| \frac{\partial \tau}{\partial r} \right| \leq \beta(\|\varphi\|).$$

These estimates are easy consequences of the properties of q and \tilde{f} . It follows that

$$|\tau(r, \varphi)| \leq \beta(\|\varphi\|)r.$$

Now we replace τ in (1.9) by $\tau(r, \varphi)$, φ by $r^{n-1}\varphi$, and divide (1.9) by r^{n-1} . Then (1.9) is reduced to

$$(1.15) \quad L\varphi + \frac{\sigma n(n-1)(n+1)}{2} \tau^i(r, \varphi) x^i + \tilde{f}(r, \tau(r, \varphi), \cdot) + r q(r, \tau(r, \varphi), \varphi) = 0.$$

By the above argument for finding $\tau(r, \varphi)$ and the properties of \tilde{f}, q we can find a unique solution $\varphi(r)$ of (1.15) for small r which lies near zero. There holds $\varphi(r) \rightarrow 0$ as $r \rightarrow 0$. Moreover, we have the following estimates:

$$\|\varphi(r)\| \leq Cr, \left\| \frac{d\varphi}{dr} \right\| \leq C$$

for a constant $C > 0$.

The uniqueness property of $\tau(r, \varphi)$ and $\varphi(r)$ can be stated precisely as follows

Proposition 1. *For each positive number C there are positive numbers $R_1 = R_1(C)$ and $R_2 = R_2(C)$ with the following properties: 1) if $Q(r, \tau, \varphi) = 0$, $0 < r \leq R_1$, $|\tau| \leq R_2$ and $\|\varphi\| \leq C$, then $\tau = \tau(r, \varphi)$; 2) if $H(r, \tau(r, \varphi), r^n \varphi) = n - \frac{\sigma n^2}{2} r^{n-1}$, $0 < r \leq R_1$ and $\|\varphi\| \leq C$, then $\varphi = \varphi(r)$.*

We omit the easy proof. (For part 2) one utilizes the equation (1.15) to show that φ is small.)

Now we consider the family of surfaces $\mathcal{F}_0 = \{\tilde{S}_r = S_{1/r, \tau(r, \varphi(r)), r^{n-1}\varphi(r)} : 0 < r \leq r_o\}$, where r_o is chosen as follows. Set $C = \sup_{r \leq R_1(1)} \|\varphi(r)\| + 1$, $r_1 = \max\{r : |\tau(r, \varphi)| \leq R_2(C) \text{ with } \|\varphi\| \leq C\}$ and then $r_o = \min\{r_1, R_1(1), R_1(C)\}$. These surfaces are diffeomorphic to S^n . By construction, \tilde{S}_r has constant mean curvature $nr - \frac{\sigma n^2}{2} r^n$. Since all the equations and known functions in the above construction are smooth, we conclude that \mathcal{F}_o is a smooth family of constant mean curvature spheres. Geometrically, we obtained this family by moving the center (the origin) of the Euclidean spheres $\partial \mathbb{B}_r(o)$ to $\tau(r, \varphi(r))/r$ and then performing the Euclidean normal perturbation $r^{n-2}\varphi(r)$. It remains to show that \mathcal{F}_o is a foliation. Put $\psi(r, x) = r^{-1}\tau(r, \varphi(r)) + r^{-1}x + r^{n-2}\varphi(r)(x)$, where $x \in S^n$. By the estimates for τ and φ we easily see that the maps $v(r, \cdot) := \psi(r, \cdot)/|\psi(r, \cdot)|$ approach the identity map from S^n to S^n in the C^2 norm as $r \rightarrow 0$. Hence $v(r, \cdot)$ is a smooth diffeomorphism for small r . We set $\tilde{\varphi}(r, x) = |\psi(r, v^{-1}(r, x))|$ where $v^{-1}(r, \cdot)$ denotes the inverse of $v(r, \cdot)$. Then \tilde{S}_r is the Euclidean normal graph of $\tilde{\varphi}(r, \cdot)$ over S^n . Employing the estimates for τ and φ we compute

$$\begin{aligned} \frac{\partial \psi}{\partial r} &= -\frac{x}{r^2} + \frac{1}{r} \frac{\partial \tau}{\partial r} + \frac{1}{r} \tau_\varphi \left(\frac{d\varphi}{dr} \right) - \frac{\tau}{r^2} + (n-2)r^{n-3}\varphi(r) + r^{n-1} \frac{d\varphi}{dr} \\ &= -\frac{1}{r^2}(x + O(r)), \\ d_x \psi &= \frac{1}{r}(I + O(r)), \quad d_x v = I + O(r), \end{aligned}$$

and finally

$$\frac{\partial \tilde{\varphi}}{\partial r} = \frac{1}{|\psi|} \psi \cdot \left(\frac{\partial \psi}{\partial r} + (d_x \psi) \left(\frac{\partial v^{-1}}{\partial r} \right) \right) = -\frac{1}{r^2} (1 + O(r)).$$

We conclude that $\tilde{\varphi}(r, x)$ is strictly decreasing w.r.t. r for r small (or better, $\tilde{\varphi}(\rho^{-1}, x)$ is strictly increasing for ρ large). Hence $\tilde{S}_r, \tilde{S}_{r'}$ are disjoint for small r, r' with $r \neq r'$. We replace r_o by a smaller positive number if necessary. Then \mathcal{F}_o constitutes a smooth foliation. It is easy to see that \mathcal{F}_o is balanced and regular at ∞ . Thus we have proven the existence part of the Main Theorem.

§2 Uniqueness.

Let Ω be the asymptotically flat end considered above and \mathcal{F}_o the constructed foliation. Let Ω_1 be the support of \mathcal{F}_o , i.e. the subdomain of Ω foliated by \mathcal{F}_o . Note that $\Omega \setminus \Omega_1$ is compact.

Definition 3. A *constant mean curvature foliation* is a codimension one C^2 foliation with closed leaves of constant mean curvature.

We have

Uniqueness Theorem. *Let \mathcal{F} be a weakly balanced and regular constant mean curvature foliation on a closed subdomain Ω' of Ω such that $\Omega \setminus \Omega'$ is compact. Then either \mathcal{F}_o is a restriction of \mathcal{F} or \mathcal{F} is a restriction of \mathcal{F}_o . In other words, \mathcal{F}_o is the unique maximal weakly balanced and regular constant mean curvature foliation in Ω_1 .*

Proof. Let \mathcal{F} be as described in the theorem. It is easy to see that the leaves of \mathcal{F} can be parametrized as a C^2 family S_t , $0 < t \leq 1$ with $S_t \neq S_{t'}$ if $t \neq t'$ and $\lim_{t \rightarrow 0} \text{diam} S_t = \infty$ (see Lemma 2.1 in [Ye1]). Then S_t lies on the interior side of $S_{t'}$, provided that $t > t'$. We put $\ell(t) = (\text{diam} S_t)^{-1}$ and $S_t^* = \alpha_{\ell(t)}(S_t)$, i.e. S_t^* is the dilation of S_t by the factor $\ell(t)$. Let $g_t = \ell(t)^2 \alpha_{1/\ell(t)}^* g$. Then g_t converges to the Euclidean metric in the C^4 topology away from the origin. Each S_t^* has constant mean curvature in the metric g_t . Because \mathcal{F} is regular, the second fundamental form of S_t^* in g_t is bounded above by a constant independent of t .

We claim that $\text{dist}(o, S_{t_k}^*)$ is uniformly bounded away from zero. Assume the contrary. Then we can find a sequence $S_{t_k}^*$ with $t_k \rightarrow 0$ which converges to an immersed surface S_∞ in $\mathbb{R}^{n+1} \setminus \{o\}$ of constant mean curvature. S_∞ has uniformly bounded second fundamental form and diameter 1. The only possible self-intersections of S_∞ are of the type that an embedded piece of S_∞ meets another from one side. A priori, S_∞ may have many components. Let S'_∞ be one component. Then $\overline{S'_\infty} \setminus S'_\infty = \{o\}$. Since S_∞ has uniformly bounded second fundamental form and constant mean curvature, the origin is a removable singularity, i.e. $\overline{S'_\infty}$ is a smooth immersed surface of constant mean curvature. Applying the classical Alexandrov reflection principle we deduce that $\overline{S'_\infty}$ is a round sphere. It follows that $\overline{S_\infty}$ is either a round sphere containing the origin or the union of two round spheres meeting tangentially at the

origin. But the condition of weak balance does not allow such limits as one readily sees. Thus we arrive at a contradiction and the claim is proven.

Consider an arbitrary sequence $S_{t_k}^*$ with $t_k \rightarrow 0$. By the above claim and the above arguments, a subsequence converges to a round sphere S_∞ of diameter 1 such that the origin lies in the open ball bounded by S_∞ . We conclude that

$$(2.1) \quad \tilde{H}(t) \text{diam} S_t = 2n + h(t)$$

for a function $h(t)$ which converges to zero as $t \rightarrow 0$, where $\tilde{H}(t)$ denotes the mean curvature of S_t (in the metric g). In particular there is for small t a unique solution $r(t)$ of the equation

$$(2.2) \quad \tilde{H}(t) = nr - \frac{\sigma n^2}{2} r^n.$$

Now we set $\hat{S}_t = \alpha_{r(t)}(\tilde{S}_t)$ and $\hat{g}_t = r(t)^2 \alpha_{1/r(t)}^* g$. Then the mean curvature $H(t)$ of \hat{S}_t in \hat{g}_t is $n - \frac{\sigma n^2}{2} r(t)^{n-1}$. By the above arguments concerning S_t^* we deduce that every sequence \hat{S}_{t_k} with $t_k \rightarrow 0$ contains a subsequence converging to a round sphere S_∞ such that the origin lies in the open ball bounded by S_∞ . Because of (2.1), $\text{diam} S_\infty = 2$. Let a_∞ denote the center of S_∞ . Then

$$|a_\infty| \leq b(\mathcal{F}) < 1.$$

We conclude that for small t every \hat{S}_t is the normal graph of a smooth function φ_t over a round sphere of radius 1 and center $a(t)$ such that $|a(t)| \leq b(\mathcal{F})$ and φ_t converges to zero in the C^4 norm as $t \rightarrow 0$. By the arguments in the proof of Lemma 2.3 in [Ye1] we can find (for small t) $v(t) \in \mathbb{R}^{n+1}$ such that $\limsup_{t \rightarrow 0} |v(t)| \leq b(\mathcal{F})$,

$\hat{S}_t = S_{1, v(t), \hat{\varphi}(t)}$ for a smooth function $\hat{\varphi}(t)$ over S^n with $\hat{\varphi}(t) \rightarrow 0$ in the C^4 norm as $t \rightarrow 0$. Moreover, the projection $P(\hat{\varphi}(t))$ vanishes. Roughly speaking, we can move the center $a(t)$ slightly to achieve that the defining function of \hat{S}_t as a graph has zero kernel component. Note that $S_t = S_{1/r(t), v(t), \hat{\varphi}(t)}$ and

$$(2.3) \quad H(r(t), v(t), \hat{\varphi}(t)) = n - \frac{\sigma n^2}{2} r(t)^{n-1}.$$

Applying (2.3), (1.8), (1.12) and the inequality $\limsup_{t \rightarrow 0} |v(t)| \leq b(\mathcal{F}) < 1$ we obtain for small t

$$(2.4) \quad \|\hat{\varphi}(t)\| \leq Cr(t)^{n-1}$$

for a constant $C > 0$. Next we estimate $v(t)$. To this end we apply (1.5) and calculate $H(r, \tau, 0)$ in a way somewhat different from (1.8). We have

$$(2.5) \quad \begin{aligned} H(r, \tau, 0)(x) = & n - \frac{n\sigma}{2} r^{n-1} \frac{1}{|x + \tau|^{n-1}} - \frac{n(n-1)\sigma}{2} r^{n-1} \frac{1}{|x + \tau|^{n+1}} \\ & - \frac{n(n-1)\sigma}{2} r^{n-1} \frac{\tau^k x^k}{|x + \tau|^{n+1}} + r^{n-1} f(r, \tau, x), \end{aligned}$$

where f is the same function as appearing in (1.8). It follows from (2.3), (2.4), (2.5) and (1.12) that

$$(2.6) \quad \left| P \left(\frac{1}{|x + v(t)|^{n-1}} + (n-1) \frac{1}{|x + v(t)|^{n+1}} + (n-1) \frac{v(t) \cdot x}{|x + v(t)|^{n+1}} \right) \right| \leq Cr(t)$$

for a positive constant C . For a fixed small t we change coordinates such that $v(t) = \ell \mathbf{e}_1$ for some $\ell \in (0, 1)$. Then

$$\begin{aligned} & - \int_{S^n} \frac{x^1}{|x + v(t)|^{n-1}} d\text{vol} - (n-1) \int_{S^n} \frac{x^1}{|x + v(t)|^{n+1}} d\text{vol} \\ & - (n-1) \int_{S^n} \frac{(v(t) \cdot x)x^1}{|x + v(t)|^{n+1}} d\text{vol} \\ & = - \int_{S^n} \frac{x^1}{|1 + \ell x^1|^{n-1}} d\text{vol} - (n-1) \int_{S^n} \frac{x^1}{|1 + \ell x^1|^n} d\text{vol} \\ & \geq (n-1) \int_{x^1 > 0} x^1 \left(\frac{1}{|1 - \ell x^1|^n} - \frac{1}{|1 + \ell x^1|^n} \right) d\text{vol} \\ & \geq 2n(n-1) \int_{x^1 > 0} \frac{(x^1)^2}{(1 + \ell x^1)^n (1 - \ell x^1)^n} d\text{vol} \\ & \geq (n-1)(n+1)\omega_{n+1}\ell, \end{aligned}$$

whence

$$(2.7) \quad |v(t)| \leq Cr(t)$$

for a constant $C > 0$.

Setting $\varphi^*(t) = r(t)^{1-n} \widehat{\varphi}(t)$ we have $Q(r(t), v(t), \varphi^*(t)) = 0$. Applying (2.4), (2.7) and Proposition 1 we deduce that $v(t) = \tau(r(t), \varphi^*(t))$ for small t . Applying (2.3) and Proposition 1 we then conclude that $\varphi^*(t) = \varphi(r(t))$ for small t . Consequently $S_t = \widetilde{S}_{r(t)}$ for small t . A continuity argument then shows that either \mathcal{F}_o is a restriction of \mathcal{F} or \mathcal{F} is a restriction of \mathcal{F}_o . \square

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